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MINIMUM VELOCITY INCREMENT
SINGLE-IMPULSE PROPULSIVE-GRAVITY
TURN WITH CONSTRAINTS ON THE
PERIAPSIS RADIUS

By Benjamin J. Garland,
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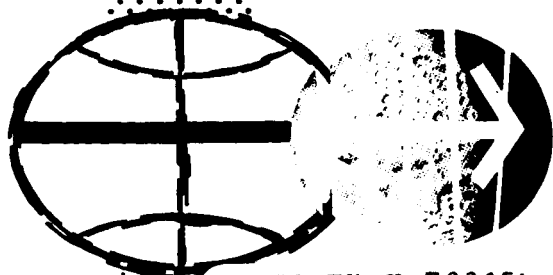
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MISSION PLANNING AND ANALYSIS DIVISION
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SUMMARY

A technique has been developed for determining the single-impulse, propulsive-gravity turn which requires the minimum velocity increment. This technique allows minimum and maximum periapsis radius to be specified. This technique is compared to a dual-impulse, propulsive-gravity turn which requires the velocity to be changed as the spacecraft enters and leaves the sphere of influence of the planet.

INTRODUCTION

The trajectory of a spacecraft will be modified significantly by the close approach to any planet. During a close approach to a planet, the gravitational attraction of the planet may be used to reduce the propulsion required to change the trajectory of the spacecraft. Normally, the trajectory of a manned spacecraft will be changed so that it will continue to another planet, which is not necessarily Earth. The trajectory of an unmanned spacecraft may be changed to achieve other conditions such as a close approach to the Sun.

It is entirely possible that the gravitational attraction of the planet may be sufficient to achieve the desired change. Examples of trajectories which can be achieved by the effect of the planet's attraction alone are presented in reference 1. These trajectories must be considered as special cases of non-stop roundtrip interplanetary trajectories.

The gravitational attraction of the planet can be supplemented by propulsion for more general cases. This type of maneuver is called a propulsive-gravity turn. The selection of the number and location of velocity impulses which result in the lowest propulsion requirement is a difficult problem. The optimum location for a single-impulse turn has been described in reference 2 and approximated in reference 3. A maneuver involving velocity changes as the spacecraft enters and leaves the sphere of influence of the planet was described in reference 4. A method of calculating the optimum number and location of impulses for this type of maneuver has been presented in reference 5.

Unfortunately, neither the method of reference 2 nor the method of reference 5 considered constraints on the periapsis radius. In fact, it was noted in reference 2 that transfers which resulted in the minimum velocity change usually take place below the surface of the planet. This paper presents a method to determine the location of a single-impulse turn which causes the impulsive-velocity change to be a minimum and keeps the periapsis radius within specified values.

SYMBOLS

A	turning angle; angle through which the spacecraft's path must be deflected while in the sphere of influence.
C_1, C_2, C_3	variables defined in equations (22a), (22b), and (22c)
e	eccentricity
f	function defined in equation (A5)
H	auxiliary angle of hyperbola
\underline{U}	velocity vector with respect to circular orbital velocity at surface of planet
U	magnitude of \underline{U}
δU	impulsive velocity increment
x	function defined by equation (20)
α	semimajor axis with respect to radius of planet
β	direction of velocity increment
γ	angle between velocity vector and local horizontal
η	true anomaly
r	radius with respect to the radius of the planet
ν	half-angle of hyperbola
τ	time from periapsis passage with respect to the period of a circular orbit at surface of planet
χ	function defined in equations (17a) and (17b)
ψ	path angle defined in figure B-1

Subscripts

c	common
I	inbound
max	maximum
min	minimum
O	outbound
p	periapsis
s	sphere of influence
T	transfer
u	unknown

ANALYSIS

The motion of the spacecraft within the sphere of influence of the target planet is described by two intersecting coplanar hyperbolas as shown in figure 1. The trajectory of the spacecraft is changed from one hyperbola to the other by an impulsive-velocity change at the intersection of the hyperbolas. The purpose of the analysis is to determine the trajectory which requires the lowest impulsive-velocity increment. The trajectory is constrained by a number of considerations which are (1) the velocity vectors as the spacecraft enters and leaves the sphere of influence of the target planet (\underline{U}_I and \underline{U}_O) must be matched, (2) the point of closest approach to the target planet must lie between some minimum and maximum value ($r_{p,min}$ and $r_{p,max}$), and (3) the time between the periapsis passage and the transfer must be greater than some specified minimum time (τ_{min}).

Basic Equations

The basic equations used in this development are standard except that they have been non-dimensionalized. The dimensional equations can be found in sources such as reference 6.

The turning angle A is determined by the velocity vectors \underline{U}_I and \underline{U}_0 . This angle is given by the equation

$$A = \cos^{-1} \left(\frac{\underline{U}_I \cdot \underline{U}_0}{U_I U_0} \right) \quad (1)$$

The semimajor axis of the hyperbola (α) can be obtained from the non-dimensional vis-viva equation which is

$$U^2 = \left(\frac{2}{1} - \frac{1}{\alpha} \right). \quad (2)$$

If the conditions at the sphere of influence are used, then

$$\alpha = \left(\frac{2}{1_s} - U_s^2 \right)^{-1}. \quad (3)$$

The eccentricity of the hyperbola (e) is

$$e = 1 - \frac{1_p}{\alpha}, \quad (4)$$

and the half-angle of the hyperbola (v) is

$$v = \cos^{-1} \left(\frac{1}{e} \right) \quad (5)$$

or

$$v = \cos^{-1} \left[\frac{1}{1 - 1_p \left(\frac{2}{1_s} - U_s^2 \right)} \right]. \quad (6)$$

The time required to travel from the periapsis is

$$\tau = \frac{1}{2\pi} \left[e \tan H - \ln \tan \left(\frac{\pi}{4} + \frac{H}{2} \right) \right], \quad (7)$$

where H is the auxiliary angle of the hyperbola defined by

$$H = 2 \tan^{-1} \left(\sqrt{\frac{e-1}{e+1}} \tan \frac{\eta}{2} \right) \quad (8)$$

and the true anomaly is

$$\eta = \cos^{-1} \left\{ \frac{1}{e} \left[\frac{\alpha(1-e^2)}{2} - 1 \right] \right\}. \quad (9)$$

The path angle at any point on the hyperbola is

$$\gamma = \tan^{-1} \left(\frac{e \sin \eta}{1 + e \cos \eta} \right). \quad (10)$$

Conditions for Gravity Turn

The turn can be accomplished by the gravitational attraction of the planet alone if $U_I = U_0$ and $r_{p,\min} \leq r_p \leq r_{p,\max}$.

The half-angle of the hyperbola, which is tangent to both \underline{U}_I and \underline{U}_0 , is

$$\nu = \frac{\pi - A}{2}. \quad (11)$$

The value of r_p can be found by substituting equation (11) into equation (6) and rearranging the resulting equation to obtain

$$r_p = \left(1 - \sqrt{\frac{2}{1 - \cos A}} \right) \left(\frac{2}{r_s} - U_s^2 \right)^{-1}. \quad (12)$$

Velocity Increment Required for Transfer at Common Periapsis

For any values of U_I , U_0 , and A , there is one combination of inbound and outbound hyperbolas which have the same periapsis. The existence of a unique common periapsis is proven in appendix A. The impulsive-velocity increment required for the transfer at the common periapsis point is usually less than 3 percent greater than the minimum (ref. 2 and 3).

The transfer can take place at the common periapsis only if $r_{p,\min} \leq r_{p,c} \leq r_{p,\max}$ and $r_{\min} = 0$.

The method for determining the common periapsis is discussed in appendix A. If the transfer can be made at the common periapsis, the velocity increment required for the propulsive-gravity turn is

$$\delta U_T = \left| \sqrt{\frac{2}{r_{p,c}} - \frac{1}{a_I}} - \sqrt{\frac{2}{r_{p,c}} - \frac{1}{a_0}} \right|. \quad (13)$$

Location of Transfer for Minimum Impulsive-Velocity Increment

If the transfer cannot be made at the common periapsis, it is necessary to find a location which will minimize the velocity change required. The location of the transfer will be specified by the periapsis radius of one hyperbola and the true anomaly measured along this hyperbola. The general model used to describe the single-impulse, propulsive-gravity turn is shown in figure 1. Figure 2 defines some of the angles which are used to describe the maneuver.

It can be seen that if

$$A + \pi = (\pi - \nu_I) + \eta_I - \eta_0 + (\pi - \nu_0)$$

or

$$A = \pi - \nu_I - \nu_0 + \eta_I - \eta_0, \quad (14)$$

only two of the variables on the right side of equation (14) are known at any time. The known quantities depend on the sign of the true anomaly of the transfer (η_T) in the following manner.

If η_T	Known quantities	Unknown quantities
>0	$\eta_I = \eta_T$ ν_I	η_0 ν_0
<0	$\eta_0 = \eta_T$ ν_0	η_I ν_I

Equation (14) can be written into two forms depending upon the sign of η_T . For $\eta_T > 0$, the form is

$$v_0 + \eta_0 = \pi - A + \eta_T - v_I, \quad (15a)$$

and for $\eta_T < 0$, it is

$$v_I - \eta_I = \pi - A - \eta_T - v_0. \quad (15b)$$

If the cosine of each side of these equations is taken and standard trigometric relations are used, the resulting equations are

$$\sin v_0 \sin \eta_0 = \cos \left(\pi - A + \eta_T - v_I \right) - \cos v_0 \cos \eta_0 \quad (16a)$$

$$- \sin v_I \sin \eta_I = \cos \left(\pi - A - \eta_T - v_0 \right) - \cos v_I \cos \eta_I. \quad (16b)$$

For convenience if $\eta_T > 0$, let

$$\chi = \cos \left(\pi - A + \eta_T - v_I \right), \quad (17a)$$

and for $\eta_T < 0$, let

$$\chi = \cos \left(\pi - A - \eta_T - v_0 \right), \quad (17b)$$

and η_u and v_u be the dependent variables. Equations (16a) and (16b) are identical if χ , η_u , and v_u are used and the equations are squared. The single resulting equation is

$$\left[\frac{\cos \eta_u}{\cos v_u} - \chi \right]^2 = (\chi^2 - 1) \left[\frac{1}{\cos^2 v_u} - 1 \right]. \quad (18)$$

Equations (4), (5), and (9) are combined to yield

$$\frac{\cos \eta_u}{\cos v_u} = \frac{x}{l_T} - 1 \quad (19a)$$

and

$$\frac{1}{\cos^2 v_u} - 1 = -\frac{x}{\alpha_u} \quad (19b)$$

where

$$x = \frac{l_{p,u}(2\alpha_u - l_{p,u})}{\alpha_u} \quad (20)$$

Equation (18) becomes

$$\frac{1}{l_T^2} x^2 + (1 + \chi) \left[\frac{1 - \chi}{\alpha_u} - \frac{2}{l_T} \right] x + (x + 1)^2 = 0 \quad (21)$$

with the aid of equations (19a) and (19b). If the variables C_1 , C_2 , and C_3 are defined as

$$C_1 = \frac{(1 + \chi) l_T^2}{2} \quad (22a)$$

$$C_2 = \frac{2}{l_T} - \frac{(1 - \chi)}{\alpha_u} \quad (22b)$$

and

$$C_3 = \sqrt{C_2^2 - \frac{4}{l_T^2}} \quad (22c)$$

then the solution for x is

$$x = C_1 (C_2 \pm C_3) \quad (23)$$

The value of $l_{p,u}$ is found to be

$$l_{p,u} = \alpha_u \left[1 \pm \sqrt{1 - \frac{x}{\alpha_u}} \right] \quad (24)$$

by solving equation (20) for $l_{p,u}$. Since $l_{p,u}$ must be positive and α_u always negative for a hyperbola, the only form of this equation which must be considered is

$$l_{p,u} = \alpha_u \left[1 - \sqrt{1 - \frac{x}{\alpha_u}} \right], \quad (25)$$

which, when equation (23) is substituted, becomes

$$l_{p,u} = \alpha_u \left[1 - \sqrt{1 - \frac{c_1}{\alpha_u} (c_2 \pm c_3)} \right]. \quad (26)$$

There are two values of $l_{p,u}$ obtained from this equation. The velocity increment required for the transfer is

$$\delta U_T = \left[U_{T,I}^2 + U_{T,O}^2 - 2U_{T,I} U_{T,O} \cos (\gamma_{T,I} - \gamma_{T,O}) \right]^{\frac{1}{2}}. \quad (27)$$

The velocity at any point along a hyperbola depends only on the semi-major axis of the hyperbola and the radius of the transfer location.

Therefore, the periapsis radii which result in the smallest change in the path angle at the transfer point will result in the smallest value of δU_T . The value of $l_{p,u}$ which is the closest to l_p will require the smallest change in the path angle.

It is possible that the spacecraft may pass through the periapsis of both the inbound and outbound hyperbolas. The spacecraft will pass through both periapsides if

$$\eta_T < 0 \quad \text{and} \quad \gamma_{T,I} > 0$$

or

$$\eta_T > 0 \quad \text{and} \quad \gamma_{T,O} < 0.$$

One of the constraints of the problem is violated if either of these conditions are true and if $i_{p,u} < i_{p,min}$. In this case, the value of i_p used to calculate $i_{p,u}$ must be increased until $i_{p,u}$ is equal to $i_{p,min}$. An estimate of the new value of i_p is found by assuming that e_u and $\eta_{T,u}$ remain constant. Therefore,

$$i_T = i_{p,u} \left(\frac{(1 + e_u)}{1 + e_u \cos \eta_{T,u}} \right). \quad (28)$$

The half-angle of the hyperbola is

$$\cos v = \left[\frac{i_p(2\alpha - i_p)}{\alpha i_T} - 1 \right]^{-1} \cos \eta_{T,u}$$

or

(29)

$$\cos v = \frac{\alpha}{\alpha - i_p}.$$

The new value of i_p is found by combining equations (28) and (29). The resulting equation is

$$i_p = \frac{1}{2} \left\{ \left(2\alpha + i_T \cos \eta_T \right) \pm \sqrt{4\alpha^2 - 4\alpha i_T + i_T^2 \cos^2 \eta_T} \right\}. \quad (30)$$

The sign of the second term in this equation is selected so that the smallest change in i_p will result. The process is repeated with the new value of i_p until $i_{p,u} = i_{p,min}$.

The direction of the velocity increment is

$$\beta_T = \tan^{-1} \left[\frac{U_{T,0} \sin \gamma_{T,0} - U_{0,I} \sin \gamma_{T,I}}{U_{T,0} \cos \gamma_{T,0} - U_{T,I} \sin \gamma_{T,I}} \right]. \quad (31)$$

The location of the transfer which results in the minimum value of δU_T is found by varying η_T until a minimum is found. The permissible value of η_T lies between the location determined by r_{min} and the point at which the hyperbola crosses the sphere of influence.

RESULTS AND DISCUSSION

The velocity increment required to turn the spacecraft through 20° is presented in figure 3 as a function of U_0 . The U_I is 1.8, and i_p is between 1.1 and 2.0. The velocity increment required by the single-impulse propulsive-gravity turn is less than that required by the dual-impulse propulsive-gravity turn of reference 4. For the dual-impulse maneuver, $\delta U_T = |U_I - U_0|$, as explained in appendix B.

Figure 4 presents the periapsis altitude for the same conditions as figure 3. The periapsis altitude of the single-impulse maneuver is 0.936 when U_0 is 1.4 and decreases to the minimum value of 0.1 when $U_0 = 2.48$. The periapsis altitude is 0.474 when $U_0 = U_I$. The periapsis altitude of the dual-impulse maneuver is 0.474 if $U_0 \leq U_I$. The periapsis altitude decreases as U_0 is increased above U_I until the minimum altitude is reached when $U_0 = 2.08$. The periapsis altitude of the dual-impulse maneuver is always less than or equal to the periapsis altitude of the single-impulse maneuver.

The effect of specifying that $i_p = 1.1$ is shown in figure 5 together with the results for the single-impulse maneuver from figure 3. The effect of restricting i_p is to increase δU_T . The minimum value of $\delta U_T = 0.12$ and occurs when $U_0 = 1.86$. (If the planet is Mars, this is 1396 fps.) The largest increase in δU_T is 0.125 and occurs when $U_0 = 1.8$.

Figure 6 illustrates what happens if the turning angle is increased to 60° . The single-impulse propulsive-gravity turn is more efficient than the dual-impulse propulsive-gravity turn if $U_0 < 1.33$. There is a discontinuity in the slope of the δU_T -versus- U_0 curve for the single-impulse maneuver when $U_0 = 1.8$. This is also the point at which the δU_T for the single-impulse maneuver exceeds δU_T for the dual-impulse maneuver by the largest amount. The δU_T for the two maneuvers approach the same value as U_0 is increased further. The periapsis radius is 1.1 over the entire range of U_0 although the maximum permissible radius is 2.0.

The location of the single-impulse maneuver for the cases presented in figure 6 is presented in figure 7 as a function of U_0 . The location of the maneuver is specified by the true anomaly and is limited by the

sphere of influence of the planet. The true anomaly of the maneuver is 2.4° when $U_0 = 0.6$ and decreases as U_0 is increased. The true anomaly continues to decrease until $U_0 = 1.33$. At this value of U_0 , the maneuver occurs as the spacecraft enters the sphere of influence. The trajectory of the spacecraft within the sphere of influence continues to change as U_0 is increased although the impulse is still applied as the spacecraft enters the sphere of influence. The impulse can be applied as the spacecraft either enters or leaves the sphere of influence if $U_0 = 1.8$. If $U_0 > 1.8$, the trajectory of the spacecraft within the sphere of influence remains the same, and the velocity is changed as the spacecraft leaves the sphere of influence.

The single-impulse, propulsive-gravity turn becomes a special case of the dual-impulse, propulsive-gravity turn whenever the impulse occurs at the sphere of influence. In this case, it is apparent that the dual-impulse, propulsive-gravity turn should require a smaller total velocity change than the single-impulse, propulsive-gravity turn. The discontinuity in the slope of the δU_T -versus- U_0 curve occurs because the location of the impulse changes from the point of entry into the sphere of influence to the point of exit from the sphere of influence as U_0 is increased above 1.8.

CONCLUDING REMARKS

A technique for determining the single-impulse, propulsive-gravity turn which requires the minimum velocity change has been developed. The periapsis radius is constrained to be within some specified range. In general, the single-impulse, propulsive-gravity turn requires a smaller velocity change than the dual-impulse, propulsive-gravity turn presented in reference 4. Under certain conditions the single-impulse maneuver degenerates into a special case of the dual-impulse maneuver. Whenever this occurs, the velocity change required by the dual-impulse maneuver is less than that required by the single-impulse maneuver.

Both the single-impulse, propulsive-gravity turn and the dual-impulse, propulsive-gravity turn should be checked if the minimum velocity change is desired. However, it should be realized that neither of these turns necessarily result in the lowest possible velocity change.

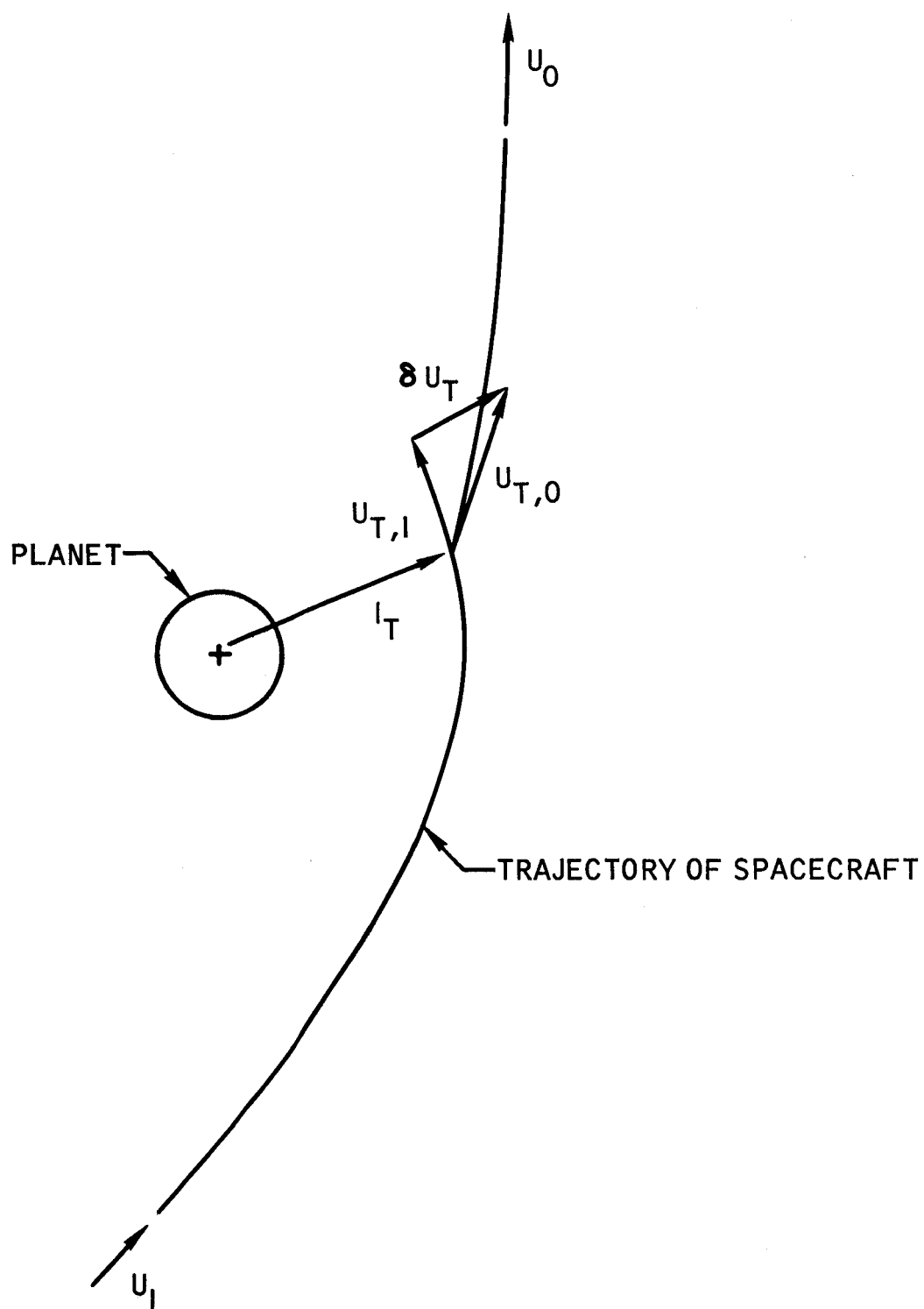


Figure 1.- Model used to describe single-impulse propulsive-gravity turn.

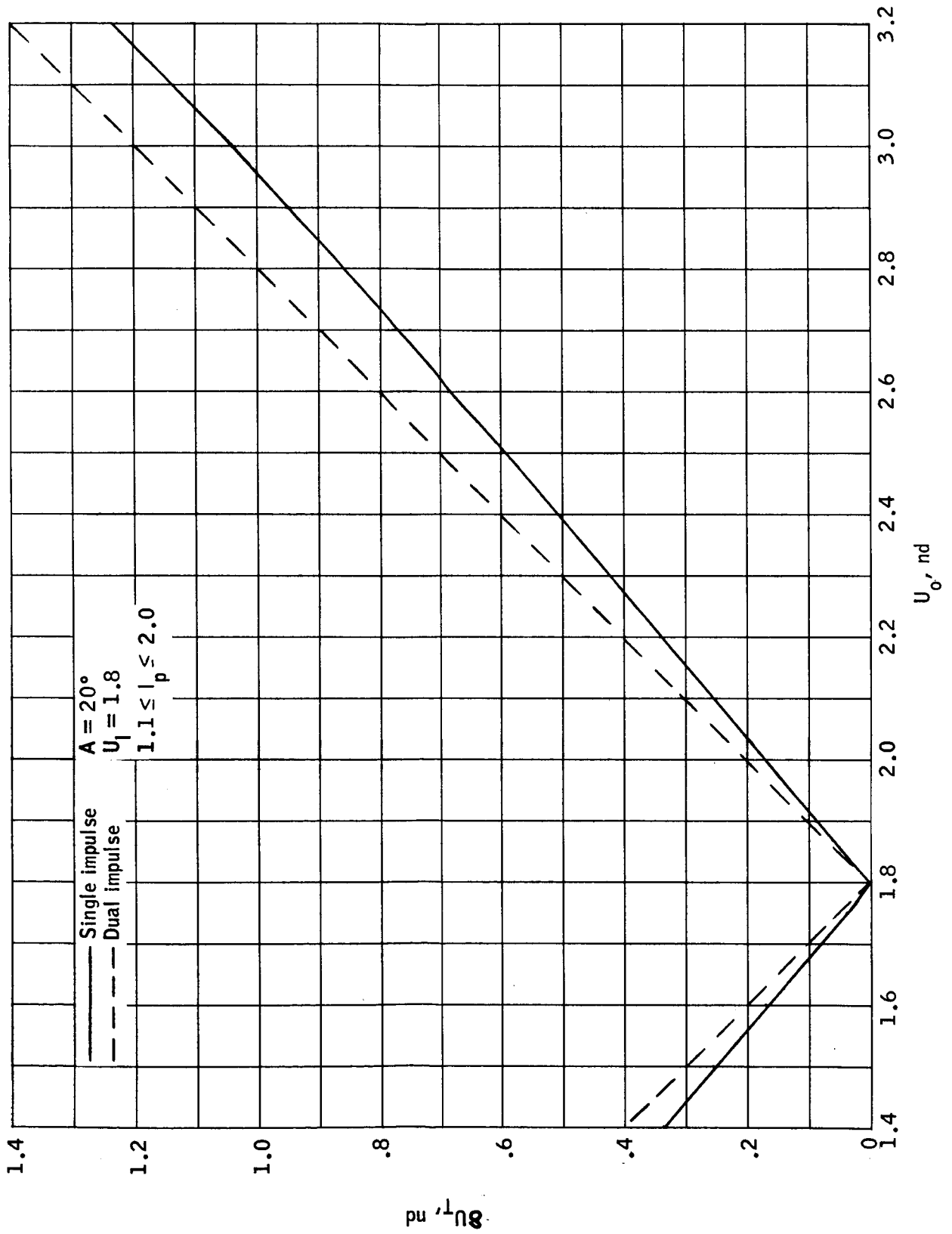


Figure 3.- Velocity increment required for propulsive-gravity turn as a function of U_0 ($A = 20^\circ$, $U_I = 1.8$).

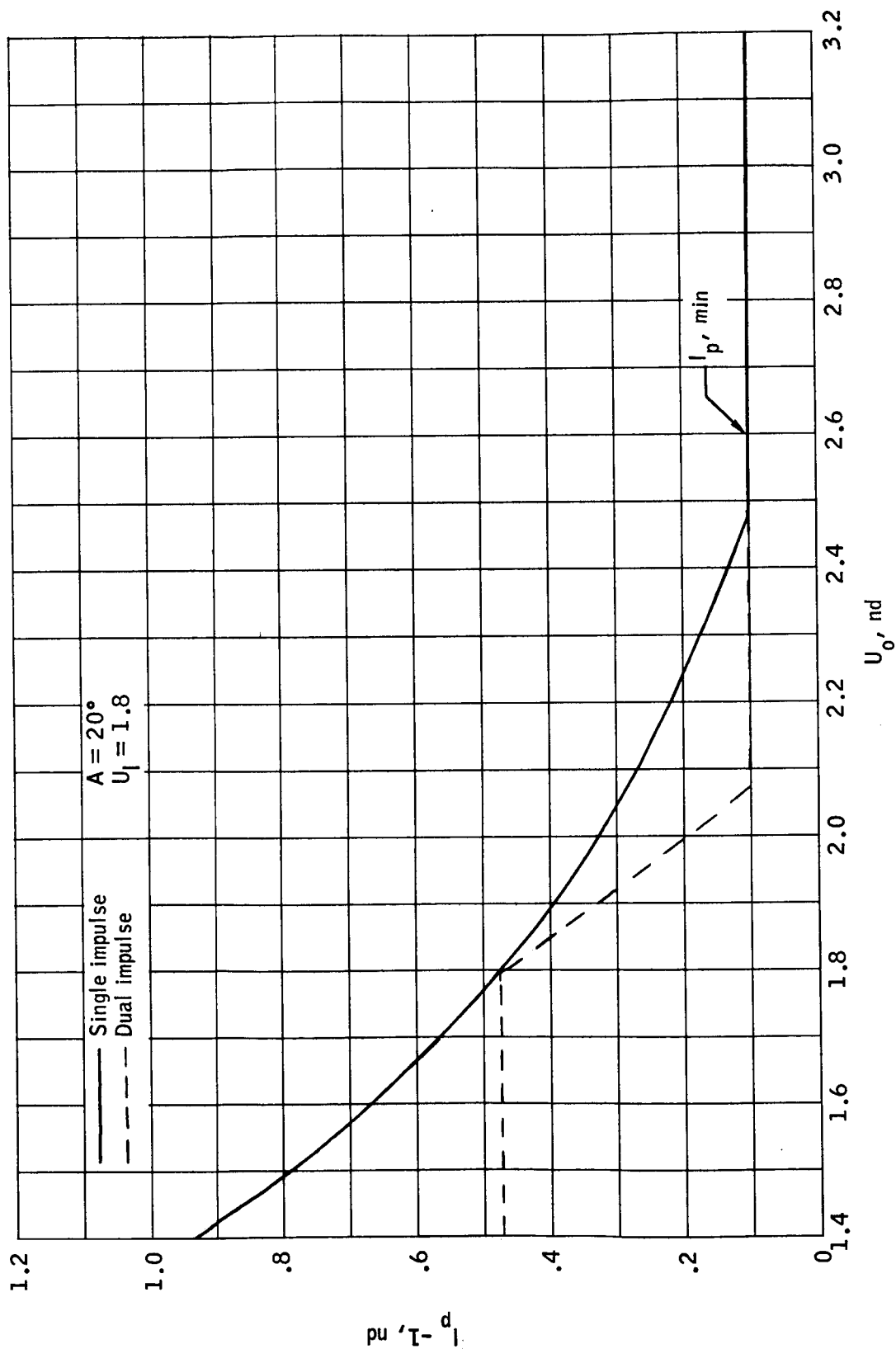


Figure 4. - Non-dimensional periapsis altitude of propulsive-gravity turn as a function of U_0 ($A = 20^\circ$, $U_l = 1.8$).

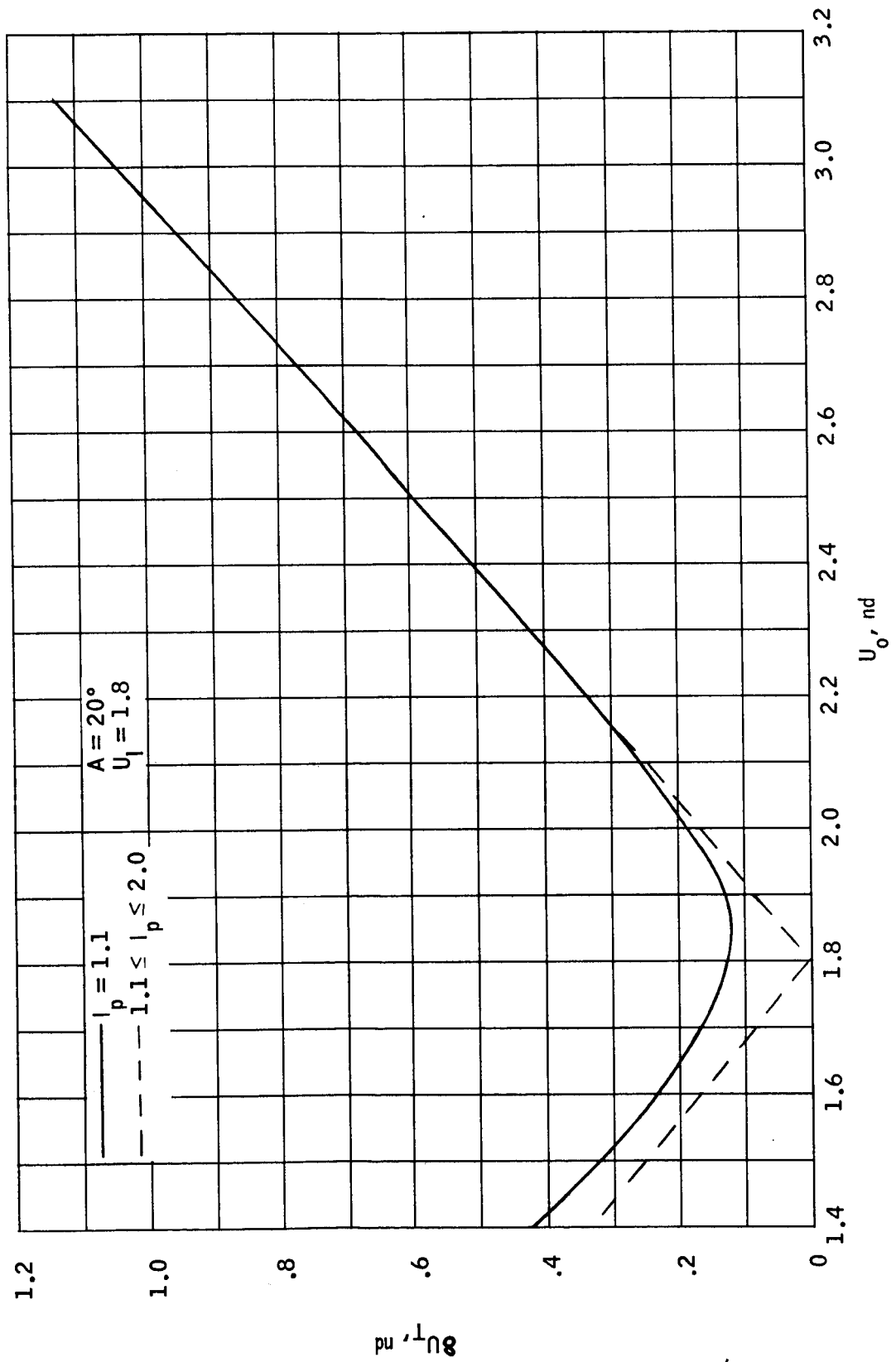


Figure 5. - Effect of specifying the periapsis radius.

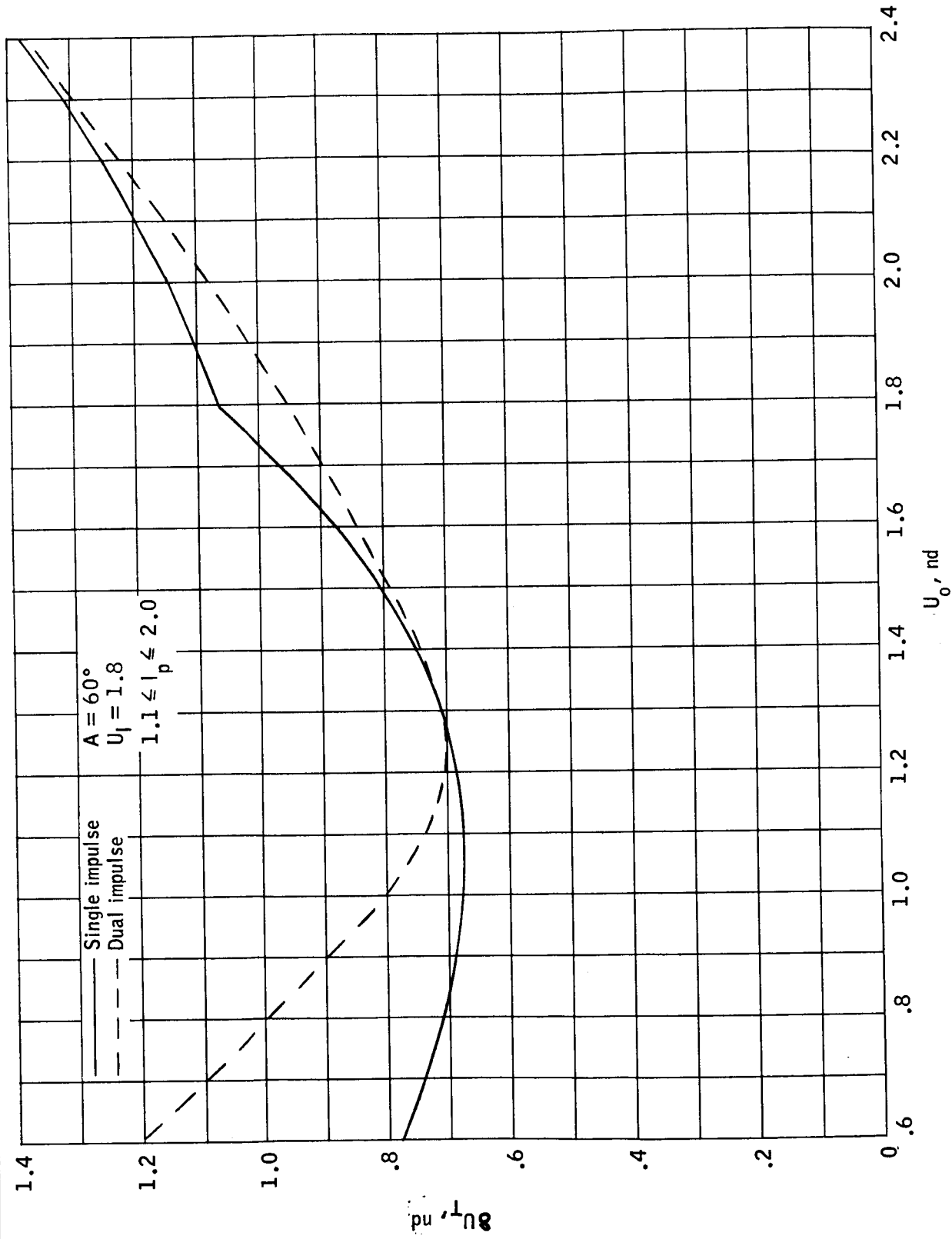
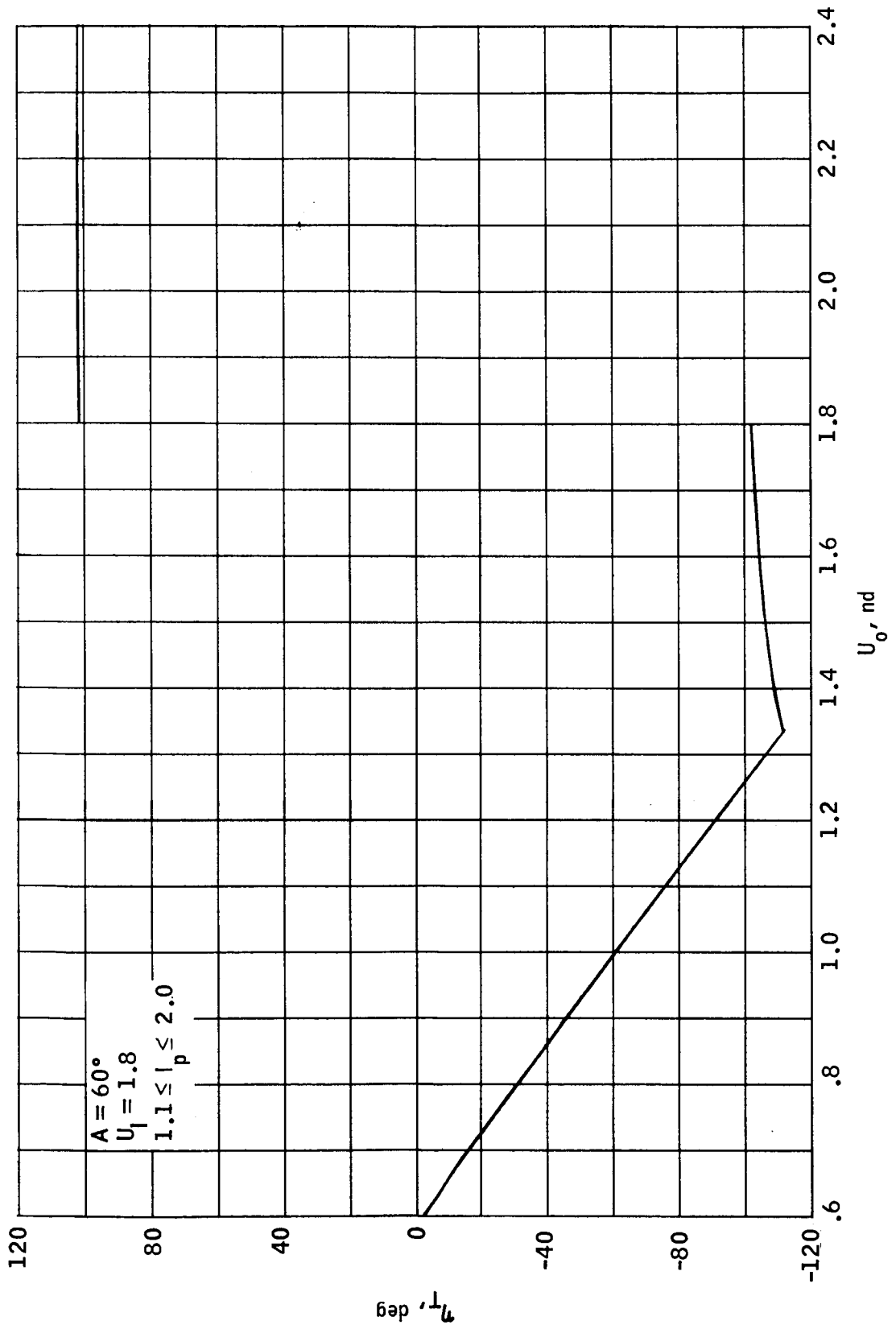


Figure 6.- Velocity increment required for propulsive-gravity turn as a function of U_0 ($A = 60^\circ$, $U_I = 1.8$).

Figure 7.- Location of velocity impulse as a function of U_0 .

APPENDIX A
DETERMINATION OF COMMON PERIAPSIS

APPENDIX A

DETERMINATION OF COMMON PERIAPSIS

The turning angle, A , is defined by the nondimensional velocity vectors \underline{U}_I and \underline{U}_O . If the transfer between the inbound hyperbola and the outbound hyperbola takes place at the periapsis of each hyperbola, then

$$\eta_{T,I} = \eta_{T,O} = 0$$

and equation (14) becomes

$$A = \pi - v_I - v_O$$

or

(A1)

$$\pi - A = v_I + v_O$$

The half-angles of the inbound and outbound hyperbolas γ_I and γ_O , are given by equation (6). These angles are found by combining equations (4). They are

$$v_I = \cos^{-1} \left(\frac{\alpha_I}{\alpha_I - r_{p,c}} \right) \quad (A2)$$

and

$$v_O = \cos^{-1} \left(\frac{\alpha_O}{\alpha_O - r_{p,c}} \right) \quad (A3)$$

The half-angle of the hyperbola is defined so that negative values have no significance. Since the semimajor axis of a hyperbola is always negative and the periapsis radius is positive, the maximum value of the half-angle is $\pi/2$. Both v_I and v_O are single-valued functions of $r_{p,c}$ and are restricted to be between 0 and $\pi/2$. Therefore,

$$0 \leq (v_I + v_O) \leq \pi$$

Since

$$0 \leq \pi - A \leq \pi$$

by definition, there is only one value of $i_{p,c}$ which will satisfy equation (A1).

If equations (A1), (A2), and (A3) are combined, the resulting equation is

$$A = \pi - \cos^{-1} \left(\frac{\alpha_I}{\alpha_I - i_{p,c}} \right) - \cos^{-1} \left(\frac{\alpha_O}{\alpha_O - i_{p,c}} \right) \quad (A4)$$

which can be solved by the Newton-Raphson technique with no danger of determining an incorrect root. The function f is defined as

$$f = A - \pi + \cos^{-1} \left(\frac{\alpha_I}{\alpha_I - i_{p,c}} \right) + \cos^{-1} \left(\frac{\alpha_O}{\alpha_O - i_{p,c}} \right) \quad (A5)$$

and its derivative with respect to $i_{p,c}$ is

$$\begin{aligned} \frac{df}{di_{p,c}} = & - \frac{\alpha_I}{(\alpha_I - i_{p,c}) \left[i_{p,c} (i_{p,c} - 2\alpha_I) \right]^{\frac{1}{2}}} \\ & - \frac{\alpha_O}{\alpha_O - i_{p,c} \left[i_{p,c} (i_{p,c} - 2\alpha_O) \right]^{\frac{1}{2}}} \end{aligned} \quad (A6)$$

A new value of i_p is given by the equation

$$(i_p)_{\text{new}} = (i_p)_{\text{old}} - \frac{f}{\frac{df}{di_{p,c}}} \quad (A7)$$

The iteration process is repeated until the value of $i_{p,c}$ is changed by less than some arbitrary amount.

APPENDIX B

DUAL-IMPULSE PROPULSIVE-GRAVITY TURN

APPENDIX B

DUAL-IMPULSE PROPULSIVE-GRAVITY TURN

A type of propulsive-gravity turn with velocity changes as the spacecraft enters and leaves the sphere of influence of the planet was described in reference 4. This maneuver is described by the model shown in figure B1. The velocity changes are given as

$$\delta U_I = \left[U_I^2 + U_S^2 - 2U_I U_S \cos(\psi_S - \psi_I) \right]^{\frac{1}{2}} \quad (B1)$$

and

$$\delta U_O = \left[U_O^2 + U_S^2 - 2U_O U_S \cos(\psi_S + \psi_u - \pi + A) \right]^{\frac{1}{2}} \quad (B2)$$

The total velocity change (δU_T) is

$$\delta U_T = \delta U_I + \delta U_O \quad (B3)$$

The δU_T is a function of the eccentricity (e), the periapsis radius (r_p) and the angle ψ_I . The condition for a minimum value of δU is

$$\Delta(\delta U_T) = \frac{\partial(\delta U_T)}{\partial e} \Delta e + \frac{\partial(\delta U_T)}{\partial r_p} \Delta r_p + \frac{\partial(\delta U_T)}{\partial \psi_I} \Delta \psi_I = 0 \quad (B4)$$

In general, this equation is valid only if the partial derivatives $\frac{\partial(\delta U_T)}{\partial e}$, $\frac{\partial(\delta U_T)}{\partial r_p}$ and $\frac{\partial(\delta U_T)}{\partial \psi_I}$ are all zero. The partial derivatives are

$$\begin{aligned} \frac{\partial(\delta U_T)}{\partial e} = & \frac{1}{2r_p U_S} \left\{ \frac{1}{\delta U_I} \left[U_S - U_I \cos(\psi_S - \psi_I) \right] \right. \\ & + \frac{1}{\delta U_O} \left[U_S - U_O \cos(\psi_S + \psi_I - \pi + A) \right] \left. \right\} \\ & + \frac{1}{e\sqrt{e^2 - 1}} \left[\frac{U_S U_I}{\delta U_I} \sin(\psi_S - \psi_I) + \frac{U_S U_O}{\delta U_O} \sin(\psi_S + \psi_I - \pi + A) \right] \end{aligned} \quad (B5)$$

$$\begin{aligned} \frac{\partial(\delta U_T)}{\partial i_p} = & \frac{e-1}{2U_s i_p^2} \left\{ \frac{1}{\delta U_I} \left[U_s - U_I \cos(\psi_s - \psi_I) \right] \right. \\ & \left. + \frac{1}{\delta U_O} \left[U_s - U_O \cos(\psi_s + \psi_I - \pi + A) \right] \right\} , \end{aligned} \quad (B6)$$

and

$$\begin{aligned} \frac{\partial(\delta U_I)}{\partial \psi_I} = & - \frac{U_s U_I}{\delta U_I} \sin(\psi_s - \psi_I) \\ & + \frac{U_s U_O}{\delta U_O} \sin(\psi_s + \psi_I - \pi + A) . \end{aligned} \quad (B7)$$

In reference 4, the value of ψ_I was found as a function of i_p and e by assuming that $|\psi_s - \psi_I|$ and $|\psi_s + \psi_I - \pi + A|$ are very small. The correct values of e and i_p were found by iterative methods.

The conditions for which the three partial derivatives given by equations (B5), (B6), and (B7) are equal to zero will now be found analytically. First, if equation (B6) is zero, then equation (B5) becomes

$$\begin{aligned} \frac{1}{e\sqrt{e^2-1}} \left[\frac{U_s U_I}{\delta U_I} \sin(\psi_s - \psi_I) + \right. \\ \left. \frac{U_s U_O}{\delta U_O} \sin(\psi_s + \psi_I - \pi + A) \right] = 0 . \end{aligned} \quad (B8)$$

A comparison of equations (B8) and (B6) shows that both can be equal to zero only if

$$\psi_s = \frac{\pi - A}{2} \quad (B9)$$

and

$$\psi_I = \frac{\pi - A}{2} . \quad (B10)$$

If equations (B1), (B2), (B9), and (B10) are substituted into equation (B6) the resulting equation is

$$\frac{U_s - U_I}{|U_s - U_I|} = - \frac{U_s - U_o}{|U_s - U_o|} \quad (B11)$$

This equation is valid only if

$$\text{sgn} (U_s - U_I) = \text{sgn} (U_s - U_o) ; \quad (B12)$$

that is, if the magnitude of U_s lies between U_I and U_o . The eccentricity of the trajectory is

$$e = \frac{1}{\cos\left(\frac{\pi - A}{2}\right)}$$

or

$$e = \frac{1}{\sin\left(\frac{A}{2}\right)} , \quad (B13)$$

and the periapsis radius is

$$r_p = \frac{e - 1}{U_s^2 - \frac{2}{r_s}} \quad (B14)$$

In summary, δU_I is a minimum if

$$\psi_I = \frac{\pi - A}{2}$$

and

$$e = \frac{1}{\sin\left(\frac{A}{2}\right)}$$

and U_s is between U_I and U_O . The minimum value of δU_T is

$$\delta U_T = |U_I - U_O|,$$

and i_p lies between $i_{p,I}$ and $i_{p,O}$.

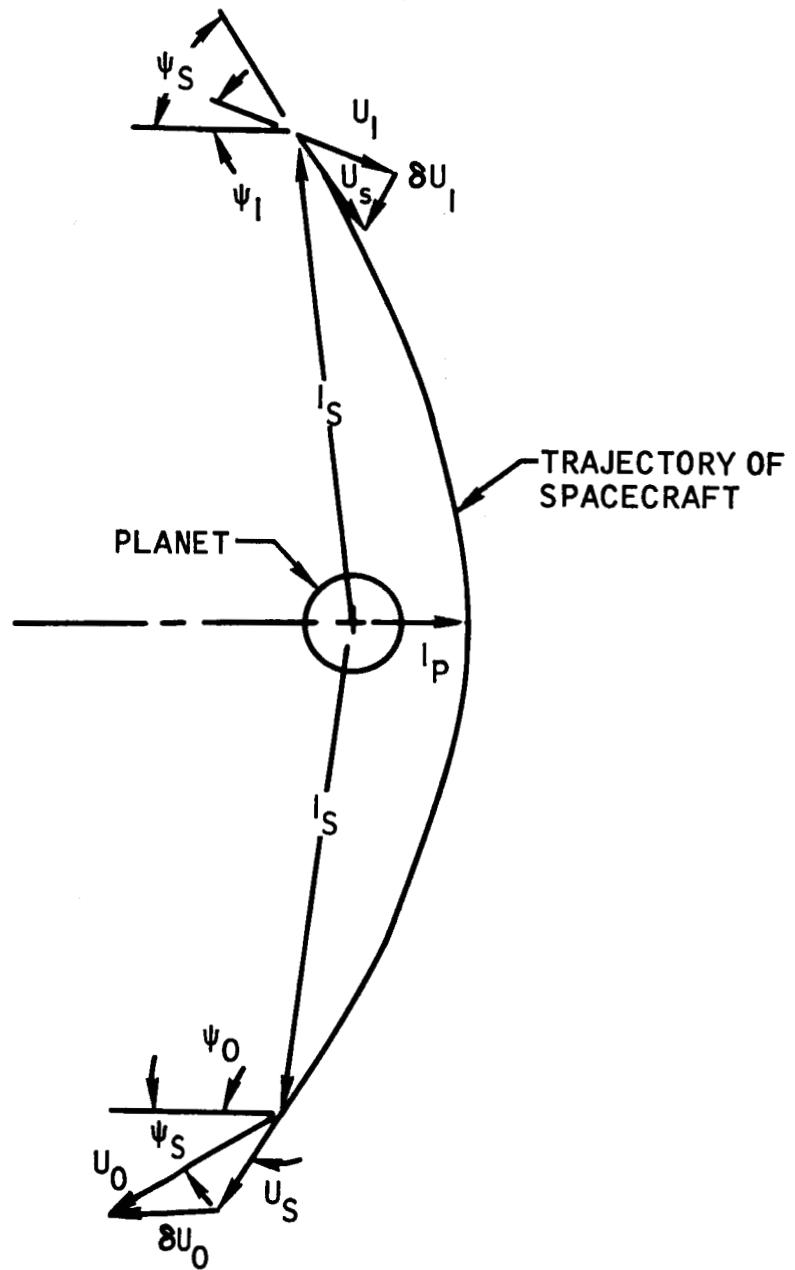


Figure B1.- Model used to describe dual-impulse powered turn.

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